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Another construction of the Virasoro group†

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Abstract. In this paper the group corresponding to the Virasoro algebra is constructed. The so-called 'descent equation' is used to derive the 2-cocycle for the group.

Recently there has been much active interest in certain infinite-dimensional Lie algebras in mathematics and theoretical physics [1]. In some ways the simplest of infinite-dimensional algebras is the Virasoro algebra. The Virasoro algebra arises in classically conformally invariant two-dimensional field theory such as the string theories of elementary particles and the theory of critical phenomena in two-dimensional statistical systems. The c-number central extension of this algebra is called the commutator anomaly of the Lie algebra of the diffeomorphism of a circle ('Schwinger-Jackiw-Johnson term'). Since the elements of the group corresponding to this Lie algebra are a finite diffeomorphism it is interesting to construct the group. This paper is a modest attempt to try to do so.

The philosophy behind this paper is to examine the Virasoro group from as many different points of view as possible. It will be obvious that the 2-cocycle for the group has been derived by Bott. Nevertheless, it is instructive to study the same problem in a new light. Our methods, on the other hand, are familiar among physicists.

Let us start with some terminology. Let M be a space with points a, and G a group of transformation with elements g and a right action on M denoted by ag. The coboundary operator Δ is defined on an n-cochain which is a function of point a and n group elements with values in some vector space in the following way [2]:

$$\begin{aligned}(\Delta\alpha_n)(a; g_1, \ldots, g_{n+1}) = {} & \alpha_n(ag_1; g_2, \ldots, g_{n+1}) - \alpha_n(a; g_1g_2, g_3, \ldots, g_{n+1}) + \ldots \\ & + (-1)^i\alpha_n(a; g_1, \ldots, g_ig_{i+1}, \ldots, g_{n+1}) + \ldots \\ & + (-1)^{n+1}\alpha_n(a; g_1, \ldots, g_n). \end{aligned} \tag{1}$$

The fundamental property of Δ is

$$\Delta^2 = 0 \tag{2}$$

which allows for the use of the cohomology technique and terminology. A cochain α_n is called a cocycle if $\Delta\alpha_n = 0$ and a coboundary if $\alpha_n = \Delta\alpha_{n-1}$ for some cochain α_{n-1}. A cocycle is non-trivial if it is not a coboundary.

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Now we choose M to be set of all Christoffel connection 1-forms, i.e. $M=\{\Gamma(x)\}$, $\Gamma^{\rho}_{\sigma}=\Gamma^{\rho}_{\sigma\mu}\,\mathrm{d}x^{\mu}$, $x\in S^n$ where S^n is compactified n spacetime, and the transformation group G to be the diffeomorphism group of spacetime S^n denoted by Diff(S^n) [3]. The product of the group is defined by the composition

$$F_1F_2(x)\equiv F_1\circ F_2(x)=F_1(F_2(x)) \tag{3}$$

where F_1, $F_2\in\mathrm{Diff}(S^n)$, $x\in S^n$, and '$\circ$' denotes composition.

The action of G on M is simply the coordinate transformation

$$\Gamma(x)\to\Gamma'(x')\equiv\Gamma F=f^{-1}(\Gamma+d)f|_x \tag{4}$$

where

$$(f^{-1})^{\alpha}_{\beta}=\partial x'^{\alpha}/\partial x^{\beta} \qquad x'=F(x) \tag{5}$$

and '$|_x$' denotes that f is evaluated at x.

The action of the diffeomorphism group admits two natural formalisms [4]: the 'passive' point of view and 'active' point of view. In what follows we shall only use the former. Then

$$\Gamma''(x'')\equiv\Gamma(F_1F_2)\equiv\Gamma(F_1\circ F_2)=(f_1(f_2)_{F_1})^{-1}(\Gamma+d)(f_1(f_2)_{F_1}) \tag{6}$$

where the subscript F_1 means that f_2 is evaluated at $F_1(x)$ when f_1 is evaluated at x.

Now let us consider Diff(S^1). Let Lie(Diff(S^1)) $\equiv$ Vect(S^1) denotes the corresponding Lie algebra. Consider the Virasoro algebra $\mathrm{Vect}(S^1)^{\wedge}=\mathrm{Vect}(S^1)\oplus \mathrm{i}\mathbb{R}$ with the commutator [5]

$$[\xi,\eta](x)=[\xi(x),\eta(x)]+\mathrm{i}\alpha\int_{S^1}(\xi'''(x)+\xi'(x))\eta(x)\,\mathrm{d}x \tag{7}$$

where ξ, $\eta\in\mathrm{Vect}(S^1)$, the bracket is the Lie bracket, and α is a real constant; $\mathrm{i}\mathbb{R}$ commutes with everything. We would like to have a construction for the group $\mathrm{Ext}(\mathrm{Diff}(S^1))\equiv\mathrm{Diff}(S^1)^{\wedge}$ which has $\mathrm{Vect}(S^1)^{\wedge}$ as its Lie algebra. Assume the composition law in the group $\mathrm{Diff}(S^1)^{\wedge}$:

$$(F_1,\lambda)(F_2,\mu)=(F_1F_2,\lambda\mu\exp(2\pi\mathrm{i}\omega(F_1,F_2))) \tag{8}$$

where λ, $\mu\in S^1$ (circle group), $F_1F_2\in\mathrm{Diff}(S^1)$ and $\omega(F_1,F_2)$ is a real-valued function of F_1 and F_2; the product F_1F_2 is defined by (3).

The problem is to compute ω. As pointed out by Segal [5], the central extension of Diff(S^1) by S^1 is topologically a product, $\mathrm{Diff}(S^1)\times S^1$, so there is no topological obstruction to the construction of a globally defined ω. Notice that this is different from the Kac-Moody group [5, 6]: the central extension of the loop group Map(S^1, SU(2)) of maps $S^1\to\mathrm{SU}(2)$ by S^1 cannot be thought of as a product $\mathrm{Map}(S^1,\mathrm{SU}(2))\times S^1$ but it should be defined as a smooth principal S^1-bundle on the infinite-dimensional manifold Map(S^1, SU(2)).

Let us consider orientation preserving diffeomorphisms, $\mathrm{Diff}_0(S^1)$. They are 'small' diffeomorphisms, i.e. they can be obtained by exponentiating vector fields. Since $\pi^2(\mathrm{Diff}(S^2))=0$ [7] the cocycle ω can be defined as follows. Let $\Omega^0_3(\Gamma)$ be the Chern-Simons form in three dimensions

$$\Omega^0_3(\Gamma)=c\mathrm{Tr}(\Gamma\,\mathrm{d}\Gamma+\tfrac{2}{3}\Gamma^3) \tag{9}$$

where c is a normalised constant. Set

$$(\Delta\Omega^0_3)(\Gamma;F)=\Omega^0_3(\Gamma F)-\Omega^0_3(\Gamma)=cd\,\mathrm{Tr}(\Gamma\,\mathrm{d}ff^{-1})-\tfrac{1}{3}c\,\mathrm{Tr}(\mathrm{d}ff^{-1})^3. \tag{10}$$

The form $c^{(3)} = -\frac{1}{3}c\,\mathrm{Tr}(\mathrm{d}ff^{-1})^3$ is closed and thus locally $c^{(3)} = \mathrm{d}H^{(2)}$ for some 2-form $H^{(2)}$. If $f = \exp(u)$, then $H^{(2)}$ can be computed from the integral [8]

$$H^{(2)}(F) = -\tfrac{1}{3}c\int_0^1 \mathrm{Tr}(\mathrm{d}f(t,x)f^{-1}(t,x))^3 \tag{11}$$

where $f(t,x) = \exp(tu(x))$. From the 'descent equation' [9]

$$(\Delta\Omega_3^0)(\Gamma;F) = \mathrm{d}\Omega_2^1(\Gamma;F) \tag{12}$$

and (10) we have

$$\Omega_2^1(\Gamma;F) = c\,\mathrm{Tr}(\Gamma\,\mathrm{d}ff^{-1}) + H^{(2)}(F). \tag{13}$$

Next we set

$$\omega(F_1,F_2) = \int_{D^2}(\Delta\Omega_2^1)(\Gamma;F_1,F_2) = \int_{D^2}\Omega_2^1(\Gamma F_1;F_2) + \Omega_2^1(\Gamma;F_1) - \Omega_2^1(\Gamma;F_1F_2)$$

$$= \int_{D^2}\lambda(F_1,F_2) \tag{14}$$

$$\lambda(F_1,F_2) = c\,\mathrm{Tr}((f_2)_{F_1}^{-1}\,\mathrm{d}(f_2)_{F_1}\,\mathrm{d}f_1f_1^{-1}) + H^{(2)}(F_1) + H^{(2)}(F_2) - H^{(2)}(F_1F_2) \tag{15}$$

where, as before, the subscript F_1 means that f_2 is evaluated at $F_1(x)$ when f_1 is evaluated at x. Notice that, although each term in (14) does depend on Γ, the function λ (thus ω) is independent of Γ which is a property of the descent equation special to two-dimensional theories. One should bear in mind that the $H^{(2)}$ are defined by the exponential representation of F_1, F_2 and F_1F_2. By the descent equation

$$(\Delta\Omega_2^1)(\Gamma;F_1,F_2) = \mathrm{d}\Omega_1^2(\Gamma;F_1,F_2) \tag{16}$$

we see that the λ in (15) can be (locally) written as a derivative of a 1-form Ω_1^2; thus, when F_1 and F_2 are sufficiently close to the unit element, the value of $\int_{D^2}\lambda(F_1,F_2)$ (thus $\omega(F_1,F_2)$) depends only on the values of F_1 and F_2 on the boundary of their domain of definition D^2 and not on their values inside the disc D^2. We shall consider the case $S^1 = \partial D^2$.

In order to show that the infinitesimal version of the function $\omega(F_1,F_2)$ we have constructed is the anomaly term in the Virasoro algebra, we have to check the cocycle condition

$$\omega(F_2,F_3) + \omega(F_1,F_2F_3) - \omega(F_1F_2,F_3) - \omega(F_1,F_2) = 0 \bmod n \tag{17}$$

which is just an expression for the associativity of the new multiplication law (8); however, it is the cocycle condition. By (16) the function $\omega(F_1,F_2)$ can be formally written as

$$\omega(F_1,F_2) = \int_{D^2}(\Delta\Omega_2^1)(\Gamma;F_1,F_2) = \int_{D^2}\mathrm{d}\Omega_1^2(\Gamma;F_1,F_2) = \int_{S^1}\Omega_1^2(\Gamma;F_1,F_2). \tag{18}$$

By a direct substitution we obtain

$$\omega(F_2,F_3) + \omega(F_1,F_2F_3) - \omega(F_1F_2,F_3) - \omega(F_1,F_2) = \int_{S^1}(\Delta\Omega_1^2)(\Gamma;F_1,F_2,F_3). \tag{19}$$

Since the Ω_1^2 can be obtained from the Ω_3^0 using the standard descent method, the analogue with [10] shows that

$$\mathbb{Z}=\int_{S^4} c\,\mathrm{Tr}\,R^2=\int_{S^2}(\Delta\Omega_2^1)(\Gamma;\,F_1,F_2)=\int_{S^1}(\Delta\Omega_1^2)(\Gamma;\,F_1,F_2,F_3) \tag{20}$$

where $R=\mathrm{d}\Gamma+\Gamma^2$ is a curvature 2-form.

We remark that, since $\pi^1(\mathrm{Diff}(S^1))=\mathbb{Z}$ [5, 7], the function $\omega(F_1, F_2)$ satisfies the cocycle condition only for orientation preserving diffeomorphisms, $\mathrm{Diff}_0(S^1)$. If the diffeomorphisms are orientation reversing, then they cannot be obtained from exponentiating vector fields (i.e. they are 'large' diffeomorphisms), and therefore the value of $\int_{D^2}\lambda(F_1, F_2)$ in (14) not only depends on the boundary values of F_1 and F_2 but also on their values inside the disc D^2. It follows that for orientation reversing diffeomorphisms the cocycle condition is not satisfied.

References

[1] Fubini S, Hanson A J and Jackiw R 1973 *Phys. Rev.* D **7** 1732
Friedan D, Qiu Z and Shenker S 1984 *Phys. Rev. Lett.* **52** 1575
Goddard P 1985 *Preprint, 85-7 DAMTP, Cambridge*
Olive D I 1985 *Preprint TP/84-85/14, Lectures given at Srni Winter School* Imperial College
Goddard P and Horsley R 1976 *Nucl. Phys.* B **111** 272
Kac V G 1980 *Adv. Math.* **35** 264

[2] Faddeev D 1984 *Phys. Lett.* **145B** 81
Zumino B 1985 *Nucl. Phys.* B **253** 477

[3] Alvarez O, Singer I M and Zumino B 1984 *Commun. Math. Phys.* **97** 409

[4] Alvarez-Gaume L and Ginsparg P 1985 *Ann. Phys., NY* **161** 423
Bardeen W A and Zumino B 1984 *Nucl. Phys.* B **244** 477

[5] Segal G 1981 *Commun. Math. Phys.* **80** 301

[6] Mickelsson J 1985 *Phys. Rev. Lett.* **55** 2099

[7] Marsden J 1974 *Applications Of Global Analysis In Mathematical Physics* (Berkeley, CA: Publish or Perish)

[8] Cronstrom C and Mickelsson J 1983 *J. Math. Phys.* **24** 2528

[9] Zumino B 1984 *Relativity, Groups and Topology* ed B S DeWitt and R Stora (Amsterdam: North-Holland)
Stora R 1983 *Cargese Lectures* (New York: Plenum)

[10] Hou B Y, Hou B Y and Wang P 1986 *Lett. Math. Phys.* **11** 179